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Flow of a Non-Newtonian Fluid past Wedges with Wall Mass Injection

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THE mathematical problem of external, boundary-layer flow of a non-Newtonian fluid was first investigated by Schowalter[1] and later by Acrivos, Shah, and Petersen.[2] Both papers established the conditions for the existence of similarity solutions for pseudoplastic, power-law-type fluids; and the latter also included numerical results for the flow over a flat plate. The extensions of the general problem to various classes of non-Newtonian, viscoelastic fluids were presented by Wells,[3] Lee and Ames,[4] and White and Metzner,[5] depending on the different models relating stress and rate of deformation tensors.

The solutions of the aforementioned investigations were obtained under the condition of impermeable walls. The effect of the wall mass injection on the flow of a non-Newtonian fluid over a flat plate was investigated by Thompson and Snyder,[6] and the solution for the stagnation flow conditions were presented by Kim and Eraslan.[7] The numerical results in both papers indicated the feasibility of further drag reduction due to the non-Newtonian characteristics of the fluid.

This Note presents the results for the boundary-layer flow of a non-Newtonian, power-law-type fluid over wedges with wall mass injection consistent with the conditions for the existence of similarity solutions for the mathematical system.

The governing equations for steady, two-dimensional flow of an incompressible fluid, under Prandtl first-order boundary-layer approximations, are

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad (1)$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = U(dU/dx) + (\partial/\partial y)(\tau_{xy}) \qquad (2)$$

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where x and y are the space coordinates along and transverse to the surface, respectively, with the associated velocity components u and v. The inviscid external velocity distribution along x, $U(x)$ is superimposed on the viscous region from the approximation of no pressure variation across the boundary layer.

For a non-Newtonian, Ostwald-de Waele, power-law-type fluid, the stress component τ_{xy} is given as

$$\tau_{xy} = \nu(\partial u/\partial y)^N \qquad (3)$$

where ν and N are constant parameters for the modified stress rate of deformation relation.

For the problem including wall mass injection, the mathematical system, Eqs. (1–3), is subject to the boundary conditions

$$u(x,0) = 0 \qquad (4a)$$

$$v(x,0) = V_w(x) \qquad (4b)$$

$$\lim_{y\to\infty} u(x,y) = U(x) \qquad (4c)$$

where $V_w(x)$ represents the injection velocity distribution on the surface.

Taking a characteristic length L along the surface, the modified form of the dimensionless Reynolds number R_e is defined in terms of the coefficient of viscosity ν and the index of the power law, N; together with the magnitude of the freestream velocity U_∞ as

$$R_e = U_\infty^{2-N}L^N/\nu \qquad (5)$$

Defining the dimensionless space coordinates and the velocity components,

$$\bar{x} = x/L \qquad \bar{u} = u/U_\infty \qquad \bar{U} = \bar{U}/U_\infty$$
$$\bar{y} = (y/L)R_e^{1/1+N} \qquad \bar{v} = (v/U_\infty)R_e^{1/1+N} \qquad (6)$$

and the dimensionless stream function $\bar{\psi}(\bar{x},\bar{y})$, which satisfies the continuity equation (1), identically,

$$\bar{u} = \partial\bar{\psi}/\partial\bar{y} \qquad \bar{v} = -\partial\bar{\psi}/\partial\bar{x} \qquad (7)$$

Substituting (6) and (7) in (2), the governing dimensionless equation for the system becomes

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2} = \bar{U}\frac{d\bar{U}}{d\bar{x}} + \frac{\partial}{\partial\bar{y}}\left(\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right)^N \qquad (8)$$

Applying the general theory of similarity via one-parameter group by Birkhoff[8] and requiring the constant conformally invariant transformation conditions, the similarity variables for the system become

$$\eta = \bar{y}\{\bar{x}^{[(N-2)p+1]/(N+1)}\}^{-1}$$
$$f(\eta) = \bar{\psi}(\bar{x},\bar{y})\{\bar{x}^{[(2N-1)]p+1]/(N+1)}\} \qquad (9)$$

where p is defined according to the conditions of existence of similarity solutions on the freestream velocity $\bar{U}(\bar{x})$ as

$$U(x) = U_\infty\bar{U}(\bar{x}) = U_\infty\bar{x}^p = U_\infty(x/L)^p \qquad (10)$$

and is directly related to the total angle between the two surfaces of the wedge.

The transformed dimensionless velocity components, in terms of $f(\eta)$, become

$$\bar{u} = \bar{x}^pf'$$
$$\bar{v} = \bar{x}^{[(2N-1)p-N]/(N+1)} \times \left[\frac{(N-2)p+1}{(N+1)}\eta f' - \frac{(2N-1)p+1}{(N+1)}\mathrm{f}\right] \qquad (11)$$

The final mathematical system for the similarity variable

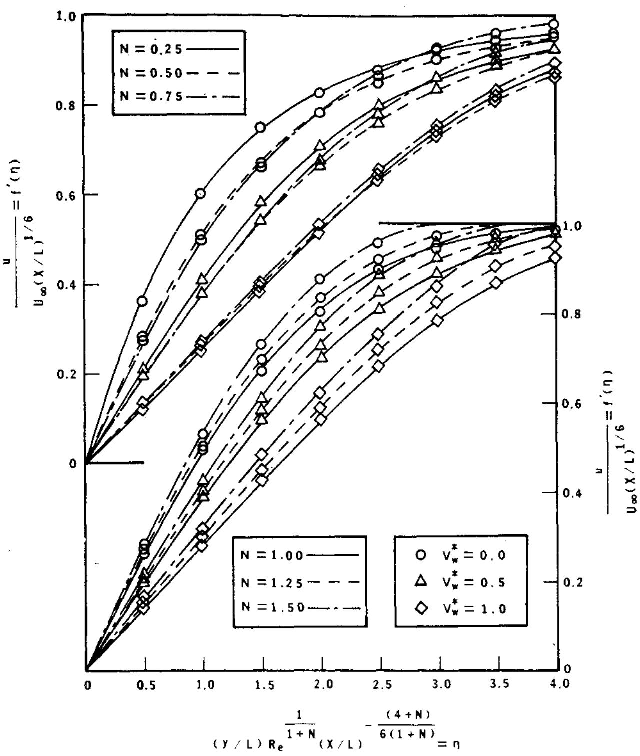

Fig. 1 Dimensionless velocity profiles for $p = \frac{1}{6}$.

$f(\eta)$ becomes

$$f''' + \left[\frac{(2N-1)p+1}{N(N+1)}\right] f(f'')^{2-N} + \frac{p}{N}(f'')^{1-N}[1-(f')^2] = 0 \quad (12)$$

$$f(0) = -V_w^* \quad (13a)$$

$$f'(0) = 0 \quad (13b)$$

$$\lim_{\eta\to\infty} f'(\eta) = 1 \quad (13c)$$

where V_w^* is a constant parameter of wall mass injection velocity, consistent with the required distribution for the existence of the similarity solutions, as

$$V_w(\bar{x}) = \{[(2N-1)p+1]/(N+1)\} \times V_w^*(x/L)^{[(2N-1)p-N]/(N+1)} \quad (14)$$

The general system (12–14) reduces to the flat plate and stagnation flow conditions for $p = 0$ and $p = 1$, respectively, and to the Newtonian fluid assumption for $N = 1$ as the limiting cases for the parameters.

The ordinary differential equation (12) was integrated by the fourth-order Runge-Kutta method with Gill coefficients, and by employing a modified version of the scheme developed by Nachtsheim and Swigert[9] for the successive iterative guesses on the initial condition $f''(0)$.

The results of the similar, dimensionless velocity profiles $f'(\eta)$ are presented in Fig. 1 and Fig. 2 for two cases of inviscid freestream flow conditions, $p = \frac{1}{6}$ and $p = \frac{1}{3}$, respectively, for various values of the wall mass injection parameter V_w^* and the non-Newtonian power-law viscosity exponent N. For $N < 1$, the figures indicate that the velocity profiles reach the freestream conditions more gradually as N increases. Consequently, the boundary-layer thickness tends to increase with N. The maximum difference in dimensionless velocity distributions between $N = 0.25$ and $N = 0.75$ values is approximately 25% for both figures, $p = \frac{1}{6}$ and $p = \frac{1}{3}$, at no wall injection case $V_w^* = 0$. For strong mass injection, $V_w^* = 1.0$, this difference is appreciably lower, with a maximum difference being approximately 8%. For $N \geq 1$, the effect of the power-law exponent N on the velocity profiles becomes considerably lower than for $N < 1$ for all cases of wall mass injection, with the maximum difference between $N = 1.0$ and $N = 1.5$ values being less than 4%.

The results clearly indicate the strong influence of wall mass injection on the velocity profiles in the boundary layer for all values of N. The increase in V_w^* results in a reduction of the steepness of the velocity profiles, hence reducing the viscous shear stress on the wedge surfaces. This effect is more pronounced for $N < 1$ (with the maximum difference between no wall injection, $V_w^* = 0$, and strong wall injection, $V_w^* = 1.0$, being approximately 60% for $N = 0.25$) than for $N > 1$ (with the maximum difference for the two corresponding wall mass injection cases being approximately 45%). The discussed effects of the power-law exponent N and the wall mass injection parameter V_w^* on the boundary-layer velocity profiles of a non-Newtonian fluid holds true for both cases of wedge flows, $p = \frac{1}{6}$ and $p = \frac{1}{3}$.

The dimensionless drag coefficient C_D is defined as

$$C_D = \frac{1}{LU_\infty^2}\int_0^L (\tau_{xy})_w dx = \frac{\nu}{LU_\infty^2}\int_0^1 \left[\frac{\partial u}{\partial y}(x,0)\right]^N dx \quad (15)$$

Substituting the dimensionless similarity variables (9) in (15) and performing the straightforward integration,

$$C_D = \frac{(N+1)}{(3Np+1)R_e^{1/(1+N)}}[f''(0)]^N$$

$$\frac{(3Np+1)R_e^{1/(1+N)}}{(N+1)} C_D = [f''(0)]^N \quad (16)$$

The variations of the scaled drag coefficient $[f''(0)]^N$ with the wall mass injection parameter V_w^* are presented in Fig. 3 parametrically for values of the power-law exponent N and the freestream wedge flow conditions p. The results indicate the strong influence of wall mass injection in the reduction of the viscous drag. For low viscous effects, $N = 0.25$,

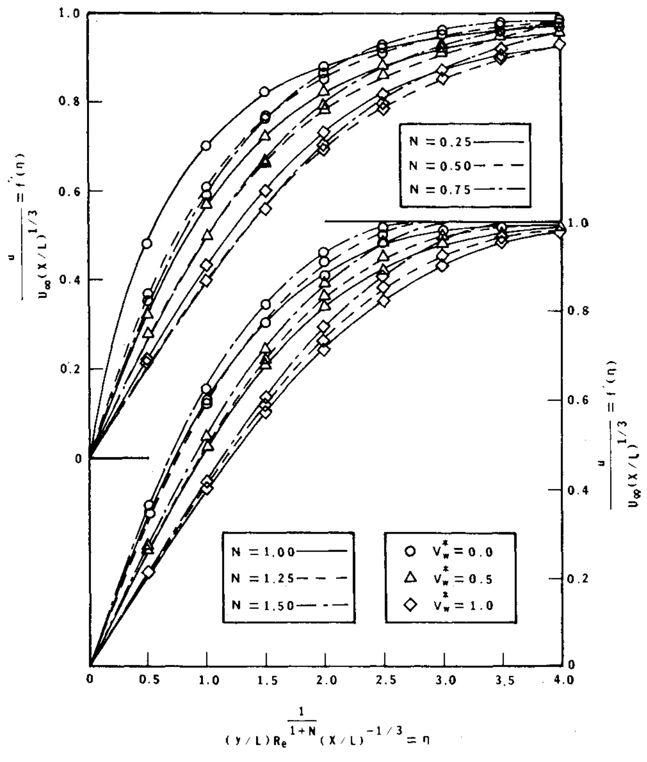

Fig. 2 Dimensionless velocity profiles for $p = \frac{1}{3}$.

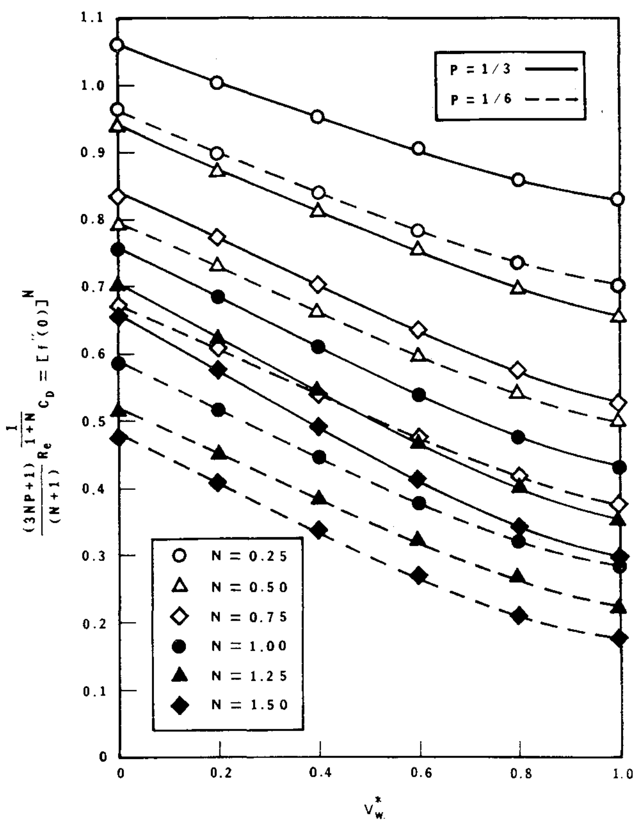

Fig. 3 Dimensionless, scaled drag coefficient.

the strong wall injection, $V_w^* = 1.0$, reduces the drag coefficient 18% for $p = \frac{1}{3}$ and 23% for $p = \frac{1}{6}$. However, for strong viscous effects, $N = 1.50$, the reduction in $[f''(0)]^N$ increases to 48% for $p = \frac{1}{3}$ and 60% for $p = \frac{1}{6}$. It is interesting to note that the variations of the scaled drag coefficient $[f''(0)]^N$ exhibit monotonically decreasing characteristics, with the negative slopes of the curves increasing with the viscous power law exponent N. However, the curves remain parallel to each other for different values of $p = \frac{1}{6}$, $\frac{1}{3}$, separated from each other by the drag coefficient at no wall injection conditions, $V_w^* = 0$, hence indicating that the effect of the wedge angle has small influence on the overall effect of the drag reduction resulting from the wall mass injection.

In summary, it is concluded that the viscous drag of the flow of a non-Newtonian fluid over wedges can be reduced considerably by the wall mass injection, and the degree of the reduction depends on the exponent of the viscous power-law model for the fluid, but not directly on the total wedge angle.

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Effects of Polymer Addition on Friction in a 10-in.-Diam Pipe

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Introduction

It has been well established that friction reductions up to 80% in turbulent liquid flows can be achieved by the addition of small amounts of certain soluble, high-molecular-weight polymers. The greatest effectiveness has been obtained using linear polymers with molecular weights of the order of several million.

Early investigators followed the assumption that the friction-reduction behavior was related to non-Newtonian fluid properties. However, later investigations proved that the friction-reduction effects become more pronounced in relatively dilute polymer concentrations, i.e., < 100 ppm, which show Newtonian behavior. Although a large number of experimental results have been reported, most of these studies have been conducted using pipes 2 in. or less in diameter. Thus, it seemed desirable to obtain some data using a pipe approximately an order of magnitude larger than any previously used.

The studies reported here were carried out in a 10-in.-diam flow facility. To our knowledge, no previous studies in facilities of this size have been reported.

Experiments conducted in large facilities have been handicapped by the difficulties associated with the preparation and injection of large quantities of solutions of friction-reducing polymers. When such polymers are added to water, the wetted particles immediately form a sticky skin, which causes the particles to clump together, rendering further solution difficult. Also, the polymers, having molecular weights of the order of 1,000,000, form extremely viscous solutions at even low concentrations. For example, a sodium salt of polyacrylamide (Separan AP30, Dow Chemical Company), which is an efficient friction reducing material, has an apparent viscosity of 5000 centipoises in a 1% solution.

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